

Quantization of Second Order Fermions

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Abstract. We review how second order equations for fields arise just by using projectors over Poincaré invariant subspaces. We focus in the case of fields describing massive spin $\frac{1}{2}$ particles, we propose a particular second order Lagrangian and present preliminary results in its quantization.

1. Introduction

A second order equation to describe massive spin $\frac{1}{2}$ particles was proposed in 1956 by Feynman and Gell-Mann [2] aiming to describe Fermi interactions. Further studies of second order QED and non-abelian gauge theories were done in [3]-[6]. These formalisms present some advantages in practical calculations versus the conventional approach [6].

Recently, a second order formalism for general fields was developed in [7] [8] [9] where the fundamental properties of theories for spin 1 and 3/2 and the corresponding electrodynamics were worked out. The spin 1/2 case has been recently presented in [10].

In this work we present preliminary results on the quantization of the spin 1/2 case.

2. Second Order Equations of Motion from the Poincaré Algebra

Second order equations arise naturally when one uses the Poincaré Casimirs to construct equations of motion for a field transforming under a given Homogeneous Lorentz Group (HLG) representation, to show this claim we turn to first principles. A local, Poincaré covariant, perturbative quantum theory is constructed out of fields $\psi(x)$ of the form

$$\psi(x) = \int d\lambda \{u(\lambda, x)a(\lambda) + v(\lambda, x)a^\dagger(\lambda)\}, \quad (1)$$

transforming as a finite dimensional representation of the HLG, where $a(\lambda)$ and $a^\dagger(\lambda)$ are the annihilation and creation operators of states in a unitary irreducible representation of the Poincaré group, so the quantum numbers λ are the two casimirs P^2 and spin W^2 , the momentum P^μ , the spin projection, the discrete symmetries and internal symmetries quantum numbers. In the case of fields transforming in representations of the HLG which contain a single irreducible representation of the Poincaré group it is possible to construct the projector over subspaces with well defined mass m and spin s using the Casimir operators P^2 and W^2 as

$$\left(\frac{P^2}{m^2}\right)\left(\frac{W^2}{-s(s+1)P^2}\right)\psi = \psi, \quad (2)$$

where $W_\alpha = \frac{1}{2}\epsilon_{\alpha\mu\nu\beta}M^{\mu\nu}P^\beta$ and therefore W^2 is quadratic in the momentum, so we can rewrite it as

$$[W^2]_{ab} = [W_\nu]_{ac}[W^\nu]_{cb} = -s(s+1)T_{ab}^{(G)\alpha\beta}P_\alpha P_\beta. \quad (3)$$

Here, the field $[\psi^{G,m,s}(x)]_a$ transforms under a representation G of the HLG and the tensor $T_{ab}^{(G)\alpha\beta}$ depends only on the generators of this representation whose internal indices are denoted by latin letters. Rearranging (2) and using (3) we get

$$\left(T_{ab}^{(G)\alpha\beta}\partial_\alpha\partial_\beta + \delta_{ab}m^2\right)\psi_b^{(G)}(x) = 0. \quad (4)$$

From this equation one can reconstruct the following Lagrangian from which the equation of motion is derived:

$$\mathcal{L} = \partial_\alpha\bar{\psi}(T^{(G)\alpha\beta} + T^{(A)\alpha\beta})\partial_\beta\psi^{(G)} - m^2\bar{\psi}\psi^{(G)}, \quad (5)$$

where $\bar{\psi}$ is some linear combination of ψ^\dagger such that the action is real and Poincaré invariant, here $T^{(A)\alpha\beta}$ is any antisymmetric constant matrix in the HLG representation space, but this matrix plays no role for the *free theory*. Note that in the derivation above only the Poincaré algebra is used, so this equation does not have restrictions related to discrete symmetries such as parity or charge conjugation.

The method described above for obtaining equations of motion using casimirs as projectors over Poincaré subspaces forms part of the NKR formalism, actually this method has been used for the description of massive spin $\frac{3}{2}$ [7], spin 1 [8] and spin $\frac{1}{2}$ [10].

3. Second Order Fermions

Next we focus on particles with mass m and spin $\frac{1}{2}$, for the Lorentz representations $G_1 = (\frac{1}{2}, 0)$, $G_2 = (0, \frac{1}{2})$ couple minimally to an electromagnetic field the expresion (4) gives

$$[(i\partial_\mu - A_\mu)^2 + \vec{\sigma} \cdot (\vec{B} \pm i\vec{E})]\psi^{G_{1,2}} = m^2\psi^{G_{1,2}}, \quad (6)$$

these are just the Feynman and Gell-Mann equations [2]. The problem with these fields $\psi^{G_{1,2}}$ is that it is not possible to create a second order Lagrangian which is both *hermitian* and Poincaré scalar using only one of these two representations $\{(\frac{1}{2}, 0), (0, \frac{1}{2})\}$. To form a *hermitian and Poincaré invariant action* one should use instead the representation $G = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, then the simplest second order Lagrangian density for this representation is found to be (here $\bar{\psi} = \psi^\dagger\beta$)

$$\mathcal{L} = \frac{1}{2}\partial_\alpha\bar{\psi}(\eta^{\alpha\beta} + \frac{g}{2}[\gamma^\alpha, \gamma^\beta])\partial_\beta\psi - \frac{m^2}{2}\bar{\psi}\psi, \quad (7)$$

where we skipped the superindex G for the sake of simplicity. The details of the calculations can be found in [10].

4. Quantization of Second Order Fermions

In the literature there are two main roads for the quantization of second order theories:

- Quantize the Dirac theory and relate the Dirac fields to those obeying the second order equations (see [4] and [5]),
- Quantize a second order Poincaré invariant *non real* action [13],

$$\mathcal{L} = \partial_\mu\psi^{\dagger(\frac{1}{2},0)}\partial^\mu\psi^{(0,\frac{1}{2})} - m^2\psi^{\dagger(\frac{1}{2},0)}\psi^{(0,\frac{1}{2})} \quad (8)$$

additionally a subsidiary condition that is not contained in the Lagrangian is imposed.

Both approaches reproduce QED results. Of course it would be nicer to quantize departing from a real action, without invoking Dirac's formalism, and using subsidiary conditions only if it is physically or mathematically necessary.

In what follows we present our preliminary results quantizing the *hermitian and Poincaré covariant theory given by (7)*. If we take ψ and ψ^\dagger as the independent Grassmann coordinates their associated moments are

$$\pi_\psi = -\frac{1}{2}(\gamma^\mu \partial_\mu \psi)^\dagger, \quad \pi_{\psi^\dagger} = \frac{1}{2}(\gamma^\mu \partial_\mu \psi), \quad (9)$$

here we are using the convention of right derivatives like in [11], then the Lagrangian doesn't have constraints and we can immediately write down the Hamiltonian

$$H = \int d^3x \{ \pi_\psi \gamma^0 \gamma^i \partial_i \psi - (\gamma^i \partial_i \psi)^\dagger \gamma^0 \pi_{\psi^\dagger} - 2\pi_\psi \gamma^0 \pi_{\psi^\dagger} + \frac{1}{2} m^2 \psi^\dagger \gamma^0 \psi \}, \quad (10)$$

this would be that of Dirac if we set $\pi_\psi = i\frac{m}{2}\psi^\dagger$!

The next step in the canonical quantization is to impose the canonical anticommutation relations, the non zero relations are

$$[\psi(\vec{x}, t), \pi_\psi(\vec{y}, t)]_+ = [\psi^\dagger(\vec{x}, t), \pi_{\psi^\dagger}(\vec{y}, t)]_+ = i\delta(\vec{x} - \vec{y}), \quad (11)$$

these non vanishing anticommutators coincide with those of the Dirac theory. Since ψ and ψ^\dagger are independent coordinates they must anticommute at equal times, i.e.

$$[\psi(\vec{x}, t), \psi^\dagger(\vec{y}, t)]_+ = 0, \quad (12)$$

this relation is not consistent since it can be proved that a massive spin $\frac{1}{2}$ field transforming under the representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ cannot anti-commute with its adjoint at equal times.

We have proved this claim using the Weinberg construction of fields [1], the anticommutator of the corresponding massive spin $\frac{1}{2}$ field transforming under the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation with its adjoint is never zero, here Weinberg uses the Parity operator but the result holds without imposing this symmetry.

Then, how can the theory given by (7) be quantized consistently? we are studying the following possibilities:

- To consider ψ and ψ^\dagger as independent variables and not establishing its relation is causing the problem in the quantization. In the Dirac formalism this works because this is a constrained theory [9].
- Another possibility is that in order to quantize the theory a constraint is necessary. In fact if we impose $\pi_\psi = i\frac{m}{2}\psi^\dagger$ as a second class constraint in the Hamiltonian, the resulting theory is equivalent to that of Dirac, but we haven't proved that this condition is indeed necessary, unique, or if there exist some condition less restrictive.

The last possibility is interesting from a theoretical point of view, because if a constraint is needed in order to quantize the theory then the necessity of this constraint would imply that a consistent quantization of massive spin $\frac{1}{2}$ needs additional symmetry (remember that constraints arises naturally from symmetries) apart from the Poincaré algebra from which these theories have been shown to arise.

This constraint could be related with the discrete symmetries of parity or charge conjugation, actually this relation appear already in the conventional formalism, the Dirac equation is closely related to parity (this relation is presented in [10]), also when passing from Dirac to Majorana charge conjugation symmetry is needed *even in the free theory*.

5. Conclusions

We have shown how the second order theories arise from the Poincaré symmetry of massive spin $\frac{1}{2}$ particles. Then we performed the canonical quantization of the theory (7), we can obtain exactly the Dirac theory adding a constraint, but it is still not clear if this condition is indeed necessary. More work is needed in order to understand the general case.

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